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al-Farghānī's Tables for Constructing Astrolabes: Their Mathematical Structure and their Importance for the History of Astronomical Instrumentation

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Introduction

al-Farghānī is well-known in Europe for his book *Elements of Astronomy*, in which he gives a non-mathematical introduction to the science of astronomy in clear, succinct and accessible language.[1] In the East, however, he was also renowned for his contribution to the science of the astrolabe. His treatise on the subject is entitled *al-Kāmil fī ṣanʿat al-asṭurlāb al-shimālī wa-'l-janūbī wa-ʿilalihā bi-'l-handasa wa-'l-ḥisāb* (Comprehensive (work) on the construction of the northern and southern astrolabe and its geometrical and computational principles).[2] In this work, he provides the reader who has acquired an intermediate level in the study of astronomy and mathematics with both theory and practical elements. He exposed the mathematical principles of the astrolabe in a systematic way, formulating the earliest known demonstration of the fundamental theorem of stereographic projection.[3]
His work contains a complete set of trigonometric tables for facilitating the drawing of astrolabic plates (declination, altitude and azimuth circles) for various terrestrial latitudes as well as the more basic data for constructing the rete with its various star-pointers. With these tables, al-Farghani inaugurated an important and lasting tradition in the field of Islamic instrument-making.[4] The aim of this paper is to

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1 See Sabra, Suter, Sezgin, V, pp. 259–260, VI, pp. 149–151, Matvievskaia & Rozenfeld, II, pp. 55–58.

2 Sezgin, VI, p. 150-151, No 2.

3 See Karpova & Sergeyeva.

4 An exhaustive survey of Islamic tables for constructing astrolabes is the subject of an unpublished paper by David A. King, on which the present paper is strongly dependent. I would like to

present a brief survey of al-Farghānī's treatise – which has never been completely published nor properly studied before[5] – and a detailed analysis of the mathematical structure and of the use of his very interesting tables for constructing astrolabes. I shall examine their relationship to the earlier set of tables by al-Khwārizmī and provide a general assessment of their influence and legacy in the Islamic literature on astronomical instrumentation.

The astrolabe, a Hellenistic invention that was known throughout the Near East, was transmitted to the Muslims in the eighth century and was immediately adopted and improved by them. Of the earliest writings in Arabic on this astronomical instrument we only have scanty information, but we can be sure that the great majority only dealt with its *use*. Yet Ptolemy's *Planisphærium* on the theory of stereographic projection has been translated into Arabic and several Muslim scholars were familiar with it. In his introduction al-Farghānī deplores the limited way with which his predecessors treated the science of the astrolabe:[6]

> For the principles of its construction and use, they only considered what they needed for the knowledge of the reasons behind its shape and the correctness of the instructions (on its use). We have got no information about whether any of them gave a demonstration of that or illustrated it in a book. People use it by imitating (their predecessors) in a mechanical way, relying simply on the fact that the results obtained with this instrument about the celestial movements coincide with those gained with the armillary sphere, or even with the results of calculation, although there is no proof of the instrument's accuracy nor reference to its principles.

And this is why, he says, he wrote a comprehensive (*kāmil*) work on the astrolabe. This treatise was apparently written in the year 856 A.D., for al-Farghānī associates the year 225 Yazdigird with his star table (on f. 20r of Ms. Berlin 5791). Moreover, he states earlier in the work (f. 14v) that the longitude of the star Capella (= *al-ʿayyūq* = α Aurigae) is 5;20° of Gemini "according to what we found by observation in our times (*fī zamāninā*), i.e. in the year 225 of Yezdigird of the month Shahrivār", that is September/October of the year 856 A.D. Furthermore he gives in Chapter 5 all his worked examples for latitude 30° (Cairo), and therefore I surmise that he wrote his work while being in Egypt.[7] Since al-Farghānī is known to have supervised at the request of the caliph al-Mutawakkil the construction of the new

thank Prof. King for his generously giving me access to this unpublished material and allowing me to make use if it here. See King, "Tables" in the bibliography. An expanded version coauthored by King and myself is in preparation.

[5] The introduction has been translated in German: see Wiedemann, "Einleitungen". I did not have access to the Russian translation of the first chapter by Sergeyeva (Moscow, 1970) mentioned by Matvievskaia & Rozenfeld nor the other Russian works mentioned in their bibliography, with the exception of the paper mentioned in footnote 3, where extracts from Chapter 1 are given.

[6] Mss. Berlin 5791 and 5790. Cf. Wiedemann "Einleitungen", pp. 22-23.

[7] Ms. Berlin 5791, ff. 49r and 56r. On f. 56v he states that the maximum and minimum values of daylight "in our climate" are approximately 14 and 10 hours, respectively; this agrees well with his value of the maximum daylight for latitude 30° (12;58 hours).

nilometer on the island of Rawḍa near al-Fusṭāṭ (Old Cairo), which was completed in the year 861/862 A.D., I consider very plausible that our astronomer had stayed several years in Egypt at that time and wrote his treatise on the astrolabe there.[8]

Mathematical preliminaries

For reasons of simplicity we shall use here modern trigonometric functions to base 1, but one should bear in mind that medieval astronomers used them with sexagesimal base 60. Sexagesimal numbers are denoted according to the convention $a;b,c$, meaning a degrees, b minutes and c seconds. All entries in al-Farghānī's tables are in standard Arabic alphanumerical notation (*abjad*).
Let us consider the trigonometrical relations involved in the stereographic projection (see **Figure 1**). The projection of any point P to the south pole will meet the plane of the equator at Q. Now it can easily be proven that the segment OQ, the distance of that point from the centre of the astrolabe, is given by the relation

$$R\,\frac{\sin NP}{1+\cos NP} = R\,\frac{\sin NP}{\operatorname{vers} NP} = \tan\frac{NP}{2}$$

where $R = 0P$ is the radius of the basic sphere. From now on we shall designate this function by f, so that $f(\theta) = R\ \tan\frac{\theta}{2}$, where $\theta = NP$.
In passing we observe two interesting identities of this function:

$$f(90+\theta)+f(90-\theta) = 2\ \sec\theta$$

and

$$f(90+\theta)-f(90-\theta) = 2\ \tan\theta.$$

The projection of the parallels of Cancer and Capricorn (**Figure 2**) will have radii of

$$f(90-\varepsilon) = R\ \tan\frac{90-\varepsilon}{2}$$

and

$$f(90+\varepsilon) = R\ \tan\frac{90+\varepsilon}{2}$$

respectively, where ε is the obliquity of the ecliptic, taken to be $23;33°$ by our author, in accordance with the value observed by al-Ma'mūn's astronomers and recorded in the *Mumtaḥan Zīj* (but see below about the duplicity of values for this parameter). al-Farghānī assumes the diameter of his astrolabic plate — whose outer rim coincides with the projected circle of Capricorn — to have 60 units, so that $R = 19;39$, which is the value of his basic radius. The radius for Cancer will in turn measure $12;52$.[9]
On **figure 3** PU represents the horizon and DE a circle of altitude h, and these circles project on QV and GF respectively. The terrestrial latitude is given by $PN = \phi$. Let us denote by r the radius of a projected circle and by d the distance of its centre

[8] On al-Farghānī's involvement with the nilometer, see Sabra and Hill.
[9] Cf. Ms. Berlin 5791, ff. 12r–13r and f. 48r (where 12;52 is erroneouly copied as 12;54).

from the centre of the plate. The projection of a circle of altitude h for latitude ϕ is expressed by the formulae:

$$r(\phi,h) = \frac{1}{2}\{f(180-\phi-h)+f(\phi-h)\}$$

and

$$d(\phi,h) = \frac{1}{2}\{f(180-\phi-h)-f(\phi-h)\}.$$

Given these two quantities it is straightforward to draw the altitude circles on the astrolabic plate.

Let us now examine the azimuth circles. The radius of the projected prime vertical (**Figure 4a**) measures

$$x(\phi) = \frac{1}{2}\{f(90-\phi)+f(90+\phi)\} = R\ \sec\phi.$$

For a circle of azimuth a, we refer to **Figure 4b**, where Z and Y are respectively the projections of the zenith and nadir. W represents the centre of the prime vertical. Since all azimuth circles pass through Z and Y, their centres must lie on the perpendicular GH to line ZY. The projection of the azimuth circle is circle $GYHZ$ with centre C. ZC and ZW make of course an angle a between each other. We trace the chords ZH and GZ, which cuts diameter IJ at L, and observe that triangles ZWH and GCL are similar, so that we get the following relations:

$$\frac{WH}{ZW} = \frac{CL}{GC} = \frac{GC\,f(90-a)}{R\,GC}.$$

Defining the auxiliary function $WH = A(\phi,a)$, we can express it as

$$A(\phi,a) = \frac{x(\phi)f(90-a)}{R} = \sec\phi f(90-a).$$

Knowing this quantity, it is not difficult to construct the centre C and thus the azimuth circle on the astrolabic plate.

Analysis of al-Farghānī's tables

The quantities tabulated by al-Farghānī are as follows:

1. Table of the function $f(\theta)$ for $1 \le \theta \le 179$;
2. Table of the right ascension $\alpha(\lambda)$ for $1 \le \lambda \le 90$;
3. Table of 25 fixed stars with coordinates $\lambda, \beta, \Delta, \alpha, \mu, f(\Delta)$ (see below for an explanation of these symbols);
4. Tables of $r(\phi, h)$ and $d(\phi, h)$, $0 \le h \le 90$, for $\phi = 0, 15-55$;
5. Table of $A(\phi, a)$ for $a = 0, 5, 10, \ldots 90$, and for the same range of ϕ.

We shall designate these five tables by **T1**, **T2**, **T3**, **T4** and **T5**. al-Farghānī was not the first Muslim scientist to compile a table of the function $f(\theta)$. A manuscript in Berlin contains a treatise on the astrolabe by al-Khwārizmī, in which a table apparented to **T1** is found (but no other tables).[10] al-Khwārizmī compiled it up to argument 90, using a basic radius of 150, a value related to the Indian tradition. al-Khwārizmī's table was not compiled with an accuracy comparable to that of al-Farghānī.

T2 and **T3** serve the construction of the rete with its ecliptic markings and star pointers. In **T2** the right ascension is accurately computed to one sexagesimal place. It is not based on the value of the obliquity of the ecliptic advocated by al-Farghānī, but rather on Ptolemy's value 23;51. It was thus copied from an earlier table. The star table **T3** is based on the *Mumtaḥan* star table as found in the Mumtaḥan *Zīj* and the *Zīj* of Ḥabash al-Ḥāsib, but adjusted to account for the movement of precession between the *Mumtaḥan* observations (in the year 214 H. = 829 A.D.) and the year of al-Farghānī's compilation 225 Y. = 856 A.D., so that star longitudes λ have been augmented of 0; 15° over the original values.[11] All other parameters, the latitude β, declination Δ, mediation μ and right ascension α are likewise taken from the same source. On the other hand the radii of the stars on the rete, given by $f(\Delta)$, are based on **T1** and have been computed by al-Farghānī himself.

To construct the altitude and azimuth circles on an astrolabic plate for any latitude within the range given, one uses the data compiled in the appropriate subtables of **T4** and **T5**. There is no need for any more tables, as other elements usually engraved on 9th-century astrolabes (hour-curves, sine quadrant or shadow square) can be constructed in a straightforward manner.

Analysis of T1.

It is clear from the discussion in the last section that **T4** and **T5** can be compiled by arithmetical combinations of the entries of **T1**. It is thus appropriate to first investigate this fundamental table of the function $f(\theta)$. Recomputation based on the

[10] See King, "al-Khwārizmī", pp. 23–26, and idem, "Tables".

[11] See Girke, where the table is edited and analyzed. Actually 20 of the 25 stars are from the *Mumtaḥan* table, the other 5 might be related to al-Farghānī's own observations.

modern formula shows considerable errors for arguments near 180. (See column 1 in Table 1, where we give al-Farghānī's entries with errors against modern recomputation, following the convention error = recomputed - tabulated value; all errors are given in *minutes*.) This makes us suspecting that al-Farghānī had not recourse to a tangent table. I have made different attempts on my computer at reconstructing this table with procedures that may have been used by al-Farghānī, and I finally solved the problem by assuming that he used a sine table (with base 60) with two sexagesimal places, that is, with degrees, minutes and seconds. Column 2 shows my recomputation based on this hypothetical sine table with the corresponding errors. The error of -408 minutes for entry 179 has now completely vanished, as did most of the errors from entry 150 onwards. There remains a few problematic entries for arguments 170, 173 and 178. These errors could be due to slightly erroneous entries in the original sine table. Reckoning backwards, we are able to deduce the following emendations to this sine table:

Sin(88): 59;57,48 → 59;57,49 [affects f(88), f(92), f(178)]
Sin(83): 59;33,10 → 59;33,11 [affects f(83), f(97), f(173)]

Among the entries of **T1** affected by these emendations, only entries 173 and 178 show significant changes, and recomputation based on our emended sine values yields:

$$f^*(178) \rightarrow 1130;42,00 \text{ [-]}$$
$$f^*(173) \rightarrow 321;29,00 \text{ [-]}$$

Both errors indeed vanish. ($f^*(x)$ denotes hypothetical original values of **T1*** based on that sine table and with two sexagesimal places; **T1** would have been obtained from **T1*** by rounding to the minutes.)
The error of entry 170 could be explained by emending two entries in the sine table as follows:

Sin(80): 59;05,18 → 59;05,19 [affects f(80), f(100), f(170)]
Sin(10): 10;25,08 → 10;25,13 [affects f(10), f(170), f(100)]

Following this, only entry 170 of **T1** would change, and recomputation yields:

$$f^*(170) \rightarrow 224;39,37 \text{ [-]}$$

The error again vanishes but both errors in the hypothetical sine table are less easy to explain. Other smaller errors of 1 or 2 minutes for entries 141 and 142, might likewise reflect the same kind of errors in the sine table, although with less certainty:

Sin(52): 47;16,50 → 57;16,20 [affects f(52), f(142), f(128)]
Sin(39): 37;45,33 → 37;44,33 [affects f(39), f(141), f(129)]

which cause the following (significant) changes in our recomputation of **T1**:

$$f^*(142) \rightarrow 55;27,57 \text{ [-]}$$
$$f^*(141) \rightarrow 57;01,48 \text{ [-]}$$

These two errors could equally be explained by erroneous transcriptions, or by computational or rounding mistakes. For example entry $f(142)$ could have been correctly computed as 57;04 by al-Farghānī and then wrongly transcribed as 57;02. This mistake would however have occured *before* the compilation of **T4**, since its corresponding entries are based on 57;02. All remaining errors cannot be explained on account of further errors in the original sine table: they would become much too large and would badly affect other (correct) entries in **T1**. We can however safely assume that they are due to transcription errors in the original table **T1*** (with two sexagesimal places), errors which would sometimes occur directly in the minutes, or errors in the seconds which then affect rounding in the wrong direction (for example entry 47: 8;32,39 becomes 8;32,19 (following a classical transcription mistake: ح لب لط → ح لب ٮط), which is then rounded to 8;32 instead of 8;33).

Analysis of T4.
Each subtable of **T4** gives the centres and radii of the projected altitude circles on an astrolabic plate for various terrestrial latitudes. In the header of each subtable the maximum value of daylight for that latitude is given. It turns out to be once again based on Ptolemy's value of the obliquity 23;51°. This set of values was simply copied by al-Farghānī from an earlier table, just as he did with the right ascension table (**T2**). We expect the subtables of **T4** to be dependent upon **T1**. Table 2 shows the tabulated entries of the subtable for $\phi = 0$, followed by our recomputation based on the entries of **T1** and "errors", i.e. differences in minutes between these two columns. Properly speaking, deviations of $\pm\frac{1}{2}'$ should not be seen as "errors", but rather as an indication of al-Farghānī's preference for rounding. Whereas most of the entries of the form $x;y,30$ have been truncated, there is a significant number of cases of upward rounding, so that we cannot assume a systematic rounding procedure: it was more arbitrary than otherwise.

A priori, we would not expect any error of $\pm 1'$ (or larger), since the only operation involved consists in adding or subtracting two entries of **T1** and dividing the result by two, yet in the subtable for latitude 0° we do find six pairs of entries $d(0,h), r(0,h)$ displaying errors of $\pm 1'$, as well as one individual error of $\frac{3}{2}$. The fact that most of these errors occur pairwise indicates that one of the two quantities x,y involved in the operation $\frac{x \pm y}{2}$ deviated by one or two minutes from the corresponding entry of **T1**. Examining the errors in the subtables for latitudes 15 and 30, one now finds only 7 and 4 errors of ± 1 respectively. The only explanation for these errors is careless execution or transcription errors. I have spotted in the subtable for $\phi = 0$ three cases of 'contamination': the minutes of quantity y has also been used as the minutes of x.

Analysis of T5.
Only the entries for $a = 0$ (i.e. $x(\phi)$) have been accurately computed from **T1** using formula $x(\phi) = \frac{1}{2}(f(90-\phi) + f(90+\phi))$. But the column for $\phi = 0$ surprisingly does not agree with corresponding entries of **T1**! $A(0,a)$ should indeed correspond

to $f(90-a)$, yet most entries are wrong. Given the fact that

$$A(\phi,a) = \frac{A(\phi,0)A(0,a)}{R},$$

it would have been a very easy task to compile the table by multiplying entries of row 1 with entries of column 1 and then dividing by R. But al-Farghānī did not realize that and appears to have computed this table using some interpolation method. No interpolation pattern is discernible, however, for the first and second differences occur quite at random. I just cannot explain how this table might have been computed. All I can say is that it does not correspond to the system employed for generating **T4** from **T1**. It is as though some incompetent had done the job in place of al-Farghānī. Notwithstanding these somewhat disparaging remarks from my part, one should bear in mind that *for practical purposes*, the errors involved have no consequence whatsoever on the accuracy of the astrolabic markings.

The influence of al-Farghānī's approach to astrolabe construction

As we said above, al-Farghānī was not the very first astronomer to compile tables intended at constructing astrolabes. al-Khwārizmī did compile a table of the radii of day-circles similar to **T1**, but apparently did not go beyond. al-Farghānī was the first scientist to provide astrolabists with a whole set of tables for constructing all the required astrolabic markings. His approach was mentioned by 10th-century astronomers of high rank, like al-Sijzī and Abū Naṣr b. ʿIrāq.[12] al-Bīrūnī mentioned him critically in his treatise on the various kinds of astrolabe in relation to special projections, but did not inform his readers of the former's tables. al-Bīrūnī did however include a table of radii similar to **T1**, as did al-Marrākushī in the 13th century in his monumental work on time-keeping and instruments.

Sets of tables with a structure similar to those of al-Farghānī occur only in the writings of various Mamluk and Ottoman astronomers (from the 13th century onwards). Around the year 1285, the Yemeni Sultan al-Ashraf compiled tables of $d(\phi,h)$ and $r(\phi,h)$ for seven latitudes between the limits 13° and 24°. In the middle of the following century in Cairo, the astronomer al-Bakhāniqī composed a work intended to be a "completion" (*tatmīm*) of al-Farghānī's work. He reproduced the latter's tables and added his own subtables of $d(\phi,h)$ and $r(\phi,h)$ for all missing latitudes, from 1° to 14°, and from 51° to 90°. His tables for the azimuth circles are only partly based on those of al-Farghānī. He gives the values of $A(\phi,0)$ for latitudes 0° to 90° (whereas corresponding entries are taken from al-Farghānī) and he also tabulated the functions $12\tan\phi$ and $12\sec\phi$ for the same range of arguments.[13] al-Bakhāniqī's star table is identical to that of al-Farghānī, and he made no attempt to correct it for his own epoch! Various later works, many of them anonymous, contain tables belonging to the al-Farghānī tradition and I will refrain to mention them here. Two of the more interesting examples are given by Ibn ʿAṭīya, who

[12] This section relies heavily upon the unpublished paper of King, "Tables", mentioned before.

[13] Ms. Dublin Chester Beatty 4090.

reproduced al-Farghānī's table **T1** and Taqī al-Dīn, the well-known founder and director of the Istanbul Observatory in the late 16th century. His tables were compiled independently of al-Farghānī's ones, but they do show a similar structure.

As a final remark, one may ask to what extent these tables were actually used for constructing astrolabes. It is not possible to give a definitive and complete answer, but one can be sure that it was common practice already in the 9th-century to compile tables for facilitating the markings of different astronomical instruments. Expert astronomers (many are known to have made astrolabes themselves), probably preferred to achieve the task this way instead of using geometrical constructions. We cannot confirm, however, how professional astrolabists proceeded. In Mamluk Egypt and Syria, to judge from the number of extant tables of al-Farghānī's tradition, the numerical approach was probably more common than the geometrical one. But we know for certain that in 17th-century Persia and India, astrolabists had always recourse to geometrical methods. Henri Michel, whose 1947 book on the astrolabe is one of the best available on the subject, described this geometrical method of the Persians, but then suggested to his readers who might want to make their own astrolabes that they should use trigonometric tables.[14] But he was unaware that already one thousand years before, Muslim astronomers had used precisely the same method, and that al-Farghānī — رحمه الله — had calculated the necessary tables for all latitudes.

[14] See Michel, p. 62.

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Appendixes

T1

θ	$f(\theta)$		$f^*(\theta)$		θ	$f(\theta)$		$f^*(\theta)$	
1	00;10	0	00;10,17	0	46	08;20	0	08;20,27	0
2	00;20	-1	00;20,35	-1	47	08;32	-1	08;32,39	-1
3	00;31	0	00;30,52	0	48	08;44	-1	08;44,55	-1
4	00;41	0	00;41,10	0	49	08;57	0	08;57,18	0
5	00;51	0	00;51,29	0	50	09;09	-1	09;09,47	-1
6	01;02	0	01;01,47	0	51	09;22	0	09;22,21	0
7	01;12	0	01;12,07	0	52	09;35	0	09;35,02	0
8	01;22	0	01;22,27	0	53	09;48	0	09;47,50	0
9	01;33	0	01;32,47	0	54	10;01	0	10;00,44	0
10	01;43	0	01;43,09	0	55	10;14	0	10;13,45	0
11	01;54	0	01;53,32	0	56	10;27	0	10;26,53	0
12	02;04	0	02;03,55	0	57	10;40	0	10;40,09	0
13	02;14	0	02;14,20	0	58	10;53	-1	10;53,32	-1
14	02;25	0	02;24,46	0	59	11;07	0	11;07,03	0
15	02;35	0	02;35,13	0	60	11;20	-1	11;20,42	-1
16	02;46	0	02;45,42	0	61	11;34	0	11;34,29	0
17	02;56	0	02;56,12	0	62	11;48	0	11;48,25	0
18	03;07	0	03;06,44	0	63	12;02	0	12;02,29	0
19	03;17	0	03;17,18	0	64	12;16	-1	12;16,43	-1
20	03;28	0	03;27,53	0	65	12;31	0	12;31,06	0
21	03;38	-1	03;38,31	-1	66	12;46	0	12;45,39	0
22	03;49	0	03;49,10	0	67	13;00	0	13;00,22	0
23	04;00	0	03;59,52	0	68	13;16	1	13;15,15	1
24	04;10	-1	04;10,36	-1	69	13;31	1	13;30,18	1
25	04;21	0	04;21,23	0	70	13;46	0	13;45,33	0
26	04;32	0	04;32,12	0	71	14;01	0	14;00,58	0
27	04;43	0	04;43,03	0	72	14;16	-1	14;16,36	-1
28	04;54	0	04;53,57	0	73	14;32	0	14;32,25	0
29	05;05	0	05;04,55	0	74	14;48	0	14;48,26	0
30	05;16	0	05;15,55	0	75	15;04	-1	15;04,41	-1
31	05;27	0	05;26,58	0	76	15;21	0	15;21,08	0
32	05;38	0	05;38,04	0	77	15;38	0	15;37,49	0
33	05;49	0	05;49,14	0	78	15;55	0	15;54,44	0
34	06;00	0	06;00,27	0	79	16;12	0	16;11,53	0
35	06;12	0	06;11,44	0	80	16;29	0	16;29,18	0
36	06;23	0	06;23,05	0	81	16;47	0	16;46,58	0
37	06;34	0	06;34,29	0	82	17;05	0	17;04,53	0
38	06;46	0	06;45,58	0	83	17;23	0	17;23,05	0
39	06;57	-1	06;57,30	0	84	17;41	-1	17;41,35	-1
40	07;09	0	07;09,07	0	85	18;00	0	18;00,21	0
41	07;21	0	07;20,49	0	86	18;19	0	18;19,26	0
42	07;33	0	07;32,35	0	87	18;39	0	18;38,50	0
43	07;44	0	07;44,25	0	88	18;59	0	18;58,33	0
44	07;56	0	07;56,21	0	89	19;19	0	19;18,36	0
45	08;08	0	08;08,21	0	90	19;39	0	19;39,00	0

θ	$f(\theta)$		$f^*(\theta)$		θ	$f(\theta)$		$f^*(\theta)$	
91	20;00	0	19;59,46	0	136	48;38	0	48;38,06	0
92	20;21	0	20;20,53	0	137	49;53	0	49;53,03	0
93	20;42	0	20;42,25	0	138	51;11	0	51;11,23	0
94	21;04	0	21;04,19	0	139	52;33	0	52;33,22	0
95	21;26	-1	21;26,39	-1	140	53;59	0	53;59,18	0
96	21;49	0	21;49,25	0	141	55;28	-1	55;29,25	-1
97	22;12	-1	22;12,37	-1	142	57;02	-2	57;04,03	-2
98	22;36	0	22;36,17	0	143	58;44	0	58;43,38	0
99	23;00	0	23;00,26	0	144	60;28	-1	60;28,38	-1
100	23;25	0	23;25,04	0	145	62;19	0	62;19,20	0
101	23;50	0	23;50,14	0	146	64;17	1	64;16,20	1
102	24;16	0	24;15,57	0	147	66;20	0	66;20,15	0
103	24;43	1	24;42,12	1	148	68;32	0	68;31,38	0
104	25;10	1	25;09,03	1	149	70;51	0	70;51,19	0
105	25;37	0	25;36,30	0	150	73;20	0	73;20,01	0
106	26;05	0	26;04,36	0	151	75;59	0	75;58,53	0
107	26;34	1	26;33,20	1	152	78;49	0	78;48,47	0
108	27;03	0	27;02,45	0	153	81;51	0	81;50,48	0
109	27;33	0	27;32,54	0	154	85;07	0	85;06,55	0
110	28;04	0	28;03,47	0	155	88;38	0	88;38,02	0
111	28;36	1	28;35,27	1	156	92;27	0	92;26,48	0
112	29;08	0	29;07,56	0	157	96;35	0	96;34,58	0
113	29;41	0	29;41,17	0	158	101;06	1	101;05,32	0
114	30;15	-1	30;15,30	0	159	106;01	0	106;01,07	0
115	30;50	-1	30;50,40	-1	160	111;26	0	111;26,37	-1
116	31;27	0	31;26,48	0	161	117;26	1	117;25,27	1
117	32;04	0	32;03,57	0	162	124;04	0	124;03,48	0
118	32;42	0	32;42,11	0	163	131;29	0	131;28,59	0
119	33;21	-1	33;21,33	-1	164	139;50	1	139;49,33	0
120	34;02	0	34;02,05	0	165	149;15	0	149;15,26	0
121	34;44	0	34;43,52	0	166	160;02	0	160;02,21	0
122	35;26	-1	35;26,59	-1	167	172;28	0	172;27,59	0
123	36;11	0	36;11,27	0	168	186;58	1	186;57,44	0
124	36;57	0	36;57,23	0	169	204;03	-1	204;03,00	0
125	37;45	0	37;44,51	0	170	224;40	4	224;34,05	6
126	38;34	0	38;33,56	0	171	249;42	1	249;42,29	0
127	39;25	0	39;24,42	0	172	281;01	1	281;01,03	0
128	40;17	0	40;17,18	0	173	321;29	13	321;17,01	12
129	41;11	-1	41;11,50	-1	174	375;02	5	375;01,40	0
130	42;08	0	42;08,23	0	175	450;02	-2	450;02,15	0
131	43;07	0	43;07,05	0	176	562;52	10	562;51,51	0
132	44;08	0	44;08,04	0	177	750;29	5	750;29,02	0
133	45;12	0	45;11,31	0	178	1130;42	297	1122;08,03	514
134	46;18	0	46;17,32	0	179	2244;52	-408	2244;51,49	0
135	47;26	0	47;26,21	0	180	—			

Subtable of T4 for $\phi = 0°$

h	$d(0,h)$	$d(0,h)$[T1]		$r(0,h)$	$r(0,h)$[T1]	
1	1122;31	1122;31,00	0	1122;21	1122;21,00	0
2	565;31	565;31,00	0	565;11	565;11,00	0
3	375;31	375;30,00	-1	375;00	374;59,00	-1
4	281;46	281;46,30	$\frac{1}{2}$	281;05	281;05,30	$\frac{1}{2}$
5	225;26	225;26,30	$\frac{1}{2}$	224;35	224;35,30	$\frac{1}{2}$
6	188;02	188;02,00	0	187;00	187;00,00	0
7	161;20	161;20,30	$\frac{1}{2}$	160;08	160;08,30	$\frac{1}{2}$
8	141;12	141;11,30	$-\frac{1}{2}$	139;50	139;49,30	$-\frac{1}{2}$
9	125;38	125;37,30	$-\frac{1}{2}$	124;05	124;04,30	$-\frac{1}{2}$
10	113;11	113;11,30	$\frac{1}{2}$	111;28	111;28,30	$\frac{1}{2}$
11	103;00	102;58,30	$-\frac{3}{2}$	101;05	101;04,30	$-\frac{1}{2}$
12	94;31	94;31,00	0	92;27	92;27,00	0
13	87;21	87;21,00	0	85;07	85;07,00	0
14	81;13	81;13,30	$\frac{1}{2}$	78;48	78;48,30	$\frac{1}{2}$
15	75;55	75;55,00	0	73;20	73;20,00	0
16	71;18	71;18,00	0	68;32	68;32,00	0
17	67;12	67;12,30	$\frac{1}{2}$	64;16	64;16,30	$\frac{1}{2}$
18	63;35	63;35,30	$\frac{1}{2}$	60;28	60;28,30	$\frac{1}{2}$
19	60;21	60;21,30	$\frac{1}{2}$	57;04	57;04,30	$\frac{1}{2}$
20	57;28	57;27,00	-1	54;00	53;59,00	-1
21	54;50	54;49,30	$-\frac{1}{2}$	51;12	51;11,30	$-\frac{1}{2}$
22	52;27	52;27,30	$\frac{1}{2}$	48;38	48;38,30	$\frac{1}{2}$
23	50;17	50;17,30	$\frac{1}{2}$	46;17	46;17,30	$\frac{1}{2}$
24	48;18	48;18,30	$\frac{1}{2}$	44;08	44;08,30	$\frac{1}{2}$
25	46;29	46;29,30	$\frac{1}{2}$	42;08	42;08,30	$\frac{1}{2}$
26	44;49	44;49,30	$\frac{1}{2}$	40;17	40;17,30	$\frac{1}{2}$
27	43;17	43;17,00	0	38;34	38;34,00	0
28	41;51	41;51,30	$\frac{1}{2}$	36;57	36;57,30	$\frac{1}{2}$
29	40;32	40;32,00	0	35;27	35;27,00	0
30	39;18	39;18,00	0	34;02	34;02,00	0
31	38;09	38;09,00	0	32;42	32;42,00	0
32	37;05	37;05,00	0	31;27	31;27,00	0
33	36;04	36;04,30	$\frac{1}{2}$	30;15	30;15,30	$\frac{1}{2}$
34	35;08	35;08,30	$\frac{1}{2}$	29;08	29;08,30	$\frac{1}{2}$
35	34;16	34;15,30	$-\frac{1}{2}$	28;04	28;03,30	$-\frac{1}{2}$
36	33;26	33;25,30	$-\frac{1}{2}$	27;03	27;02,30	$-\frac{1}{2}$
37	32;38	32;39,00	1	26;04	26;05,00	1
38	31;54	31;54,00	0	25;08	25;08,00	0
39	31;12	31;12,30	$\frac{1}{2}$	24;15	24;15,30	$\frac{1}{2}$
40	30;34	30;34,00	0	23;25	23;25,00	0
41	29;57	29;57,00	0	22;36	22;36,00	0
42	29;22	29;22,00	0	21;49	21;49,00	0
43	28;48	28;48,30	$\frac{1}{2}$	21;04	21;04,30	$\frac{1}{2}$
44	28;17	28;17,00	0	20;21	20;21,00	0
45	27;47	27;47,00	0	19;39	19;39,00	0

h	$d(0,h)$	$d(0,h)$[T1]		$r(0,h)$	$r(0,h)$[T1]	
46	27;19	27;19,00	0	18;59	18;59,00	0
47	26;52	26;52,00	0	18;20	18;20,00	0
48	26;26	26;26,00	0	17;42	17;42,00	0
49	26;02	26;02,00	0	17;05	17;05,00	0
50	25;38	25;38,30	$\frac{1}{2}$	16;29	16;29,30	$\frac{1}{2}$
51	25;16	25;16,30	$\frac{1}{2}$	15;54	15;54,30	$\frac{1}{2}$
52	24;56	24;56,00	0	15;21	15;21,00	0
53	24;36	24;36,30	$\frac{1}{2}$	14;48	14;48,30	$\frac{1}{2}$
54	24;17	24;17,30	$\frac{1}{2}$	14;16	14;16,30	$\frac{1}{2}$
55	23;59	23;59,30	$\frac{1}{2}$	13;45	13;45,30	$\frac{1}{2}$
56	23;42	23;42,00	0	13;15	13;15,00	0
57	23;25	23;25,30	$\frac{1}{2}$	12;45	12;45,30	$\frac{1}{2}$
58	23;09	23;09,30	$\frac{1}{2}$	12;16	12;16,30	$\frac{1}{2}$
59	22;55	22;55,30	$\frac{1}{2}$	11;48	11;48,30	$\frac{1}{2}$
60	22;41	22;41,00	0	11;21	11;21,00	0
61	22;27	22;27,30	$\frac{1}{2}$	10;54	10;53,30	$-\frac{1}{2}$
62	22;15	22;15,00	0	10;27	10;27,00	0
63	22;03	22;03,00	0	10;01	10;01,00	0
64	21;52	21;51,30	$-\frac{1}{2}$	09;35	09;35,30	$\frac{1}{2}$
65	21;41	21;40,30	$-\frac{1}{2}$	09;10	09;09,30	$-\frac{1}{2}$
66	21;31	21;30,30	$-\frac{1}{2}$	08;45	08;44,30	$-\frac{1}{2}$
67	21;21	21;20,30	$-\frac{1}{2}$	08;20	08;20,30	$\frac{1}{2}$
68	21;12	21;12,00	0	07;56	07;56,00	0
69	21;03	21;03,30	$\frac{1}{2}$	07;32	07;32,30	$\frac{1}{2}$
70	20;55	20;55,00	0	07;09	07;09,00	0
71	20;47	20;47,00	0	06;46	06;46,00	0
72	20;39	20;39,30	$\frac{1}{2}$	06;23	06;23,30	$\frac{1}{2}$
73	20;32	20;33,00	1	06;00	06;01,00	1
74	20;26	20;26,30	$\frac{1}{2}$	05;38	05;38,30	$\frac{1}{2}$
75	20;20	20;20,30	$\frac{1}{2}$	05;16	05;16,30	$\frac{1}{2}$
76	20;15	20;15,30	$\frac{1}{2}$	04;54	04;54,30	$\frac{1}{2}$
77	20;10	20;10,30	$\frac{1}{2}$	04;32	04;32,30	$\frac{1}{2}$
78	20;05	20;05,30	$\frac{1}{2}$	04;10	04;10,30	$\frac{1}{2}$
79	20;00	20;01,00	1	03;48	03;49,00	1
80	19;56	19;57,00	1	03;27	03;28,00	1
81	19;53	19;53,30	$\frac{1}{2}$	03;06	03;06,30	$\frac{1}{2}$
82	19;50	19;50,30	$\frac{1}{2}$	02;45	02;45,30	$\frac{1}{2}$
83	19;47	19;47,30	$\frac{1}{2}$	02;24	02;24,30	$\frac{1}{2}$
84	19;45	19;45,00	0	02;04	02;04,00	0
85	19;43	19;43,00	0	01;43	01;43,00	0
86	19;42	19;41,30	$-\frac{1}{2}$	01;23	01;22,30	$-\frac{1}{2}$
87	19;41	19;40,30	$-\frac{1}{2}$	01;02	01;01,30	$-\frac{1}{2}$
88	19;40	19;40,00	0	00;41	00;41,00	0
89	19;40	19;39,30	$-\frac{1}{2}$	00;21	00;20,30	$-\frac{1}{2}$
90	19;39	19;39,00	0	00;00	00;00,00	0

Extracts from T5

a	$A(0,a)$		$A(15,a)$		$A(20,a)$		$A(25,a)$		$A(30,a)$	
0	19;39	0	20;20	0	20;55	0	21;41	0	22;41	0
5	18;05	-5	18;35	3	19;05	5	19;47	4	20;45	2
10	16;35	-6	17;00	4	17;30	3	18;07	4	19;00	2
15	15;10	-6	15;35	1	16;03	-1	16;37	0	17;23	1
20	13;47	-1	14;15	0	14;40	-1	15;10	1	15;52	2
25	12;30	1	12;58	-1	13;20	-1	13;48	0	14;27	0
30	11;20	0	11;45	-1	12;06	-2	12;31	-1	13;06	-1
35	10;14	0	10;36	0	10;55	-1	11;18	-1	11;48	1
40	09;10	-1	09;30	-2	09;46	-2	10;08	-2	10;35	-1
45	08;10	-2	08;26	-1	08;40	-1	09;00	-2	09;20	3
50	07;12	-3	07;25	-1	07;37	0	07;55	-2	08;15	0
55	06;15	-3	06;26	-1	06;36	0	06;51	-1	07;08	1
60	05;18	-2	05;28	-1	05;37	-1	05;48	1	06;03	2
65	04;22	-1	04;31	-1	04;39	-1	04;48	0	05;01	0
70	03;28	0	03;35	0	03;42	-1	03;49	0	04;00	0
75	02;36	-1	02;40	0	02;46	-1	02;51	0	03;00	-1
80	01;44	-1	01;46	1	01;50	0	01;54	0	02;00	-1
85	00;52	-1	00;53	0	00;55	-1	00;58	-2	01;00	-1

a	$A(35,a)$		$A(40,a)$		$A(45,a)$		$A(50,a)$	
0	24;00	0	25;38	0	27;47	0	30;34	0
5	21;58	1	23;30	-1	25;20	7	28;00	0
10	20;10	-2	21;34	-3	23;18	0	25;40	-2
15	18;23	1	19;44	-4	21;18	0	23;30	-4
20	16;51	-2	18;00	-2	19;24	4	21;26	-1
25	15;19	-2	16;21	-1	17;42	0	19;27	1
30	13;52	-2	14;48	-1	16;03	-2	17;36	2
35	12;30	0	13;22	-1	14;28	0	15;54	1
40	11;12	-2	11;59	-3	12;57	-1	14;15	-1
45	09;57	-1	10;38	-1	11;30	0	12;40	-1
50	08;44	0	09;20	0	10;06	1	11;07	0
55	07;34	0	08;05	0	08;46	0	09;38	1
60	06;25	1	06;53	-1	07;27	0	08;11	1
65	05;19	0	05;41	0	06;10	-1	06;46	0
70	04;14	0	04;32	-1	04;55	-1	05;23	1
75	03;10	-1	03;22	0	03;40	-1	04;02	-1
80	02;06	0	02;14	0	02;26	0	02;41	-1
85	01;03	-1	01;07	0	01;13	-1	01;20	-1